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TECHNICAL REPORT 2075

THE INFLUENCE OF THE SURFACE CONTOUR OF AN EXPLODING BODY ON FRAGMENT DISTRIBUTION

BY

WILLARD R. BENSON

OCTOBER 1954

SAMUEL FELTMAN AMMUNITION LABORATORIES
PICATINNY ARSENAL
DOVER. N. J.

ORDNANCE PROJECT TB1-0004

DEPT. OF THE ARMY PROJECT 572-01-004

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Approved:

JOHN D. ARMITAGE
Col, Ord Corps
Director
Samuel Feltman
Ammunition Laboratories

54AA   65047

TABLE OF CONTENTS

OBJECT

To determine the influence of the surface contour of an exploding body on fragment distribution.

ABSTRACT

The effect of the surface contour of an aerial exploding body upon fragment distribution is derived for a surface of revolution of arbitrary shape.

CONCLUSION

The density of the fragment distribution at any position on a target area can be determined for any given symmetrical surface contour of a missile. Generally, this distribution will be a variable.

The primary interest in desiring an analytical expression for fragment distribution is to determine its effect on lethal area. Naturally, the most desirable contour would be that which gives the maximum lethal area for a given number of fragments.

From the equations derived in this report, it is possible to determine the fragment distribution of a missile, given the equation or coordinates of the missile surface. If the fragment distribution is known, the lethal area for the particular tactical application intended for the missile can be calculated by using the information furnished in various Ballistic Research Laboratories (BRL) reports. In many cases, the computations could be set up in tabular form, so that a computer could carry out the calculations without understanding the physics of the problem.

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SYMBOLS

x - Variable distance along x axis

y - Variable distance along y axis

r - Distance from origin to element ds

α - Angle between r and the y axis

b - Distance from x axis to the center of the radius of curvature
    for any element ds

ρ - Radius of curvature for any element ds

θ - Angle between ρ and the y axis

λ - Distance from x axis to the lowest point of the body

h - Vertical distance from the lowest point of the body to the
    target surface T

a - $\lambda \neq$ h

ds- An infinitesimal arc length

d - Distance from the y axis to the x-intercept of the curve s

η - Distance from the y axis to any arbitrary point on the
    target surface

$$(-ay' \neq yy' \neq x)$$

D - Diameter of the desired target area

OP- Distance from the center of curvature of any element ds to
    point a on the target surface (see Fig 1)

As- Surface area of the exploding body

A - Projected area of the fragment

μ - Number of fragments per unit of the exploding body

K - Density of the fragment distribution - number of fragments
    per unit area impinging on the target surface

W - Weight of a fragment

y^1 - dy/dx

y^{11} - d^2y/dx^2

B - $(1 \neq (y^1)^2 - ay^{11} \neq yy^{11})$

R - Distances fragment travels from missile surface to target

T - Target surface

P - $\frac{dy}{dx}$

INTRODUCTION

1. The expected number of fragment hits on a target depends on the fragment density at the target and the projected area offered by the target. Whether the target is incapacitated or not depends on the mass, velocity, and shape of the fragments, in addition to the number and location of hits. This report will be concerned with only one of these factors, the fragment distribution. Since the most important factor in controlling fragment distribution is the surface contour of the exploding body, the analytical approach to the effect of the surface contour is considered, restricted by the following assumptions:

      a. A constant number of fragments per unit surface area of the exploding body.

      b. The fragments leave normal to the surface of the body

      c. The velocities of the fragments are of such a high value that the influence of gravity can be neglected

      d. The exploding body is considered stationary; that is, the body velocity is considered to be zero.

In order to obtain a simple mathematical solution to the problem, certain assumptions must be made. It is believed that the above assumptions are justifiable.

2. The first assumption, of course, can be controlled in the fabrication of the projectile.

3. The second assumption is certainly the most debatable. The location of the point of detonation will undoubtedly affect the normal radiation to some extent. If this influence is constant for a given design, however, as it should be, the fragment distribution, considering normal departure, should be susceptible to mathematical correction. All other influences on normal departure from a static body should be of a random nature and hence should fluctuate about the normal as an average.

4. The third assumption restricts the fragment paths to straight lines.

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5. The fourth assumption is related to the second assumption in that any velocity of the body will affect the normal radiation of the fragments. In many cases, however, the velocity of the body will have little effect and can be neglected.

6. The mathematics of this paper were completed before the author realized that parts of it paralleled analytical work accomplished at the Ballistic Research Laboratories. The surface discussed in paragraphs 13-16 of this report is known as the Kent-Hitchcock contour.*

7. In addition, H. K. Weiss ** of the Ballistic Research Laboratories has derived the optimum contour for certain tactical situations.

---

* Kent, R. H., "The Shape of a fragmentation Bomb to Produce Uniform Fragment Densities on the Ground," BRL Report No. 762

** Weiss, Herbert K., "Optimum Angular Fragment Distributions for Air-Ground Warheads," BRL Report 829

DISCUSSION OF RESULTS

General Solution for Axis Perpendicular to Target Surface

    8. General solution:
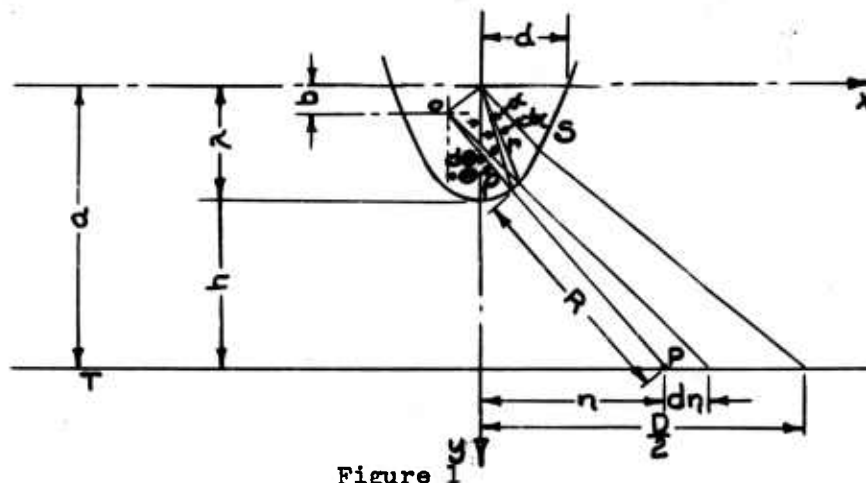
Figure 1

Consider that a missile of arbitrary shape, symmetrical about its axis, is detonated with its axis perpendicular to a plane target surface (see Fig 1). The density of the fragment distribution on the target surface T for any position η is given by

$$K = \frac{\mu\, dA_s}{dA\eta} \tag{1}$$

where

$$dA_s = 2\pi x\, ds \tag{2}$$

and

$$dA\eta = 2\pi\eta\, d\eta \tag{3}$$

Substituting Equations 2 and 3 in Equation 1 gives

$$K = \frac{\mu\, x\, ds}{\eta\, d\eta} \tag{4}$$

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9. It now remains to find ds, η, and dη as functions of x, y, y^1 and y^{11}

From Figure 1, it can be seen that

$$\eta = (R + \rho) \sin \theta - (\rho \sin \theta - r \sin \alpha) \qquad (5)$$

and that

$$(R + \rho) = \frac{a - b}{\cos \theta} \qquad (6)$$

Substituting Equation 6 in Equation 5, there results

$$\eta = (a - b) \tan \theta - \rho \sin \theta + r \sin \alpha \qquad (7)$$

It is also noted that: $\quad b = r \cos \alpha - \rho \cos \theta \qquad (8)$

Substituting Equation 8 in Equation 7, there results

$$\eta = a \tan \theta - r \cos \alpha \tan \theta + r \sin \alpha \qquad (9)$$

From elementary calculus the expression for the radius of curvature in rectangular coordinates is given by

$$\rho = - \frac{(1 + (y^1)^2)^{3/2}}{y^{11}} \qquad (10)$$

Also from Figure 1, it is seen that

$$\frac{-dy}{dx} = \tan \theta \qquad (11)$$

Therefore,

$$\sin \theta = - \frac{y^1}{(1 + (y^1)^2)^{1/2}} \qquad (12)$$

10. By using Equations 10, 11, and 12, and since $y = r \cos \alpha$ and $x = r \sin \alpha$, Equation 9 can be written as

$$\eta = - ay^1 + yy^1 + x \qquad (13)$$

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The following shows how $d\eta$ varies with x, y and y^1

$$d\eta = \frac{\partial \eta}{\partial x}\, dx + \frac{\partial \eta}{\partial y}\, dy + \frac{\partial \eta}{\partial y^1}\, dy^1 \qquad (14)$$

or

$$d\eta = \frac{\partial \eta}{\partial x}\, dx + \frac{\partial \eta}{\partial y}\, y^1 dx + \frac{\partial \eta}{\partial y^1}\, y^{11} dx \qquad (15)$$

Considering Equation 13, Equation 15 becomes

$$d\eta = (1 + (y^1)^2 - ay^{11} + yy^{11})\, dx \qquad (16)$$

Furthermore, ds can be expressed as

$$ds = (1 + (y^1)2)^{1/2}\, dx \qquad (17)$$

Finally, substituting Equations 13, 16, and 17 in Equation 4

there results

$$K = \frac{M x\, (1 + (y^1)^2)^{1/2}}{[x - (a-y)\, y^1]\, [1 + (y^1)^2 - (a-y)y^{11}]} \qquad (18)$$

From Equations 13 and 18, the density of the fragment distribution on target T can be obtained at any position η for any given surface contour of the exploding body.

11. The expression for the fragment distribution (K_n) normal to its path R at point p can be expressed simply as:

$$K_n = \frac{K}{\cos \theta} \qquad (19)$$

or since $\cos \theta = \dfrac{1}{(1 + (y^1)2)^{1/2}}$

$$\qquad (20)$$

and considering Equation 18

$$K_n = \frac{M x\, (1 + (y^1)^2)}{(-ay^1 + yy^1 + x)(1 + (y^1)2 - ay^{11} + yy^{11})} \qquad (21)$$

whereas $\eta = -ay^1 + yy^1 + x \qquad (22)$

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Simplification of General Solution

12. Although the solution obtained in the previous section is mathematically correct, in many cases it can be expressed in a form that is much simpler and nearly as accurate. This can be done by stating that :

$$R >> \rho$$
$$R >> r \qquad (23)$$

therefore $\eta = (R) \sin \theta = a \tan \theta = a y^1$ (24)

and

$$d\eta = -a y^{11} dx \qquad (25)$$

As a result, Equation 4 can be written

$$K = \frac{Mx(1 + (y^1)^2)^{1/2}}{a^2 y^1 y^{11}} \qquad (26)$$

also $K_\eta = \frac{Mx(1 + (y^1)^2)}{a^2 y^1 y^{11}}$ (27)

Where in both cases $\eta = -a y^1$ (28)

This solution would no longer be justifiable for any surface for which $\rho \longrightarrow \infty$, such as a plane or a cone.

Exact Solution for the Surface that Produces Constant Fragment Density

13. In some cases, the most desirable surface is one that produces a fragment distribution of constant density when radiating upon the target area T^*. The ideal approach to this problem would be to formally solve the nonlinear differential Equation 18, treating K as a constant. This equation of course is satisfied by the surface y = constant. There should, however, be another solution that would

* See Appendix A

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result in a curved surface, the surface in which we are interested. This can be obtained by solving the approximate Equation 26, treating K as a constant, and keeping in mind the restrictions imposed upon this equation**. The order of approximation of the solution of Equation 26 can be obtained by substituting the derived surface in Equation 18, thus obtaining the exact density of the fragment distribution produced by this surface. Equation 26 can be written as

$$\frac{K}{\pi} a^2 \, p \, \frac{dp}{dx} = x \, (1 + p^2)^{1/2} \tag{29}$$

where $p = \frac{dy}{dx}$

Transforming $\quad \frac{ka^2}{2\pi} \int (1 + p^2)^{-1/2} \, (2p \, dp) \quad = \int x \, dx \tag{30}$

Upon integrating $\quad \frac{ka^2}{\pi} \, (1 + p^2)^{1/2} \quad = \frac{x^2}{2} + C \tag{31}$

14. Now consider the boundary condition

$$p \Big)_{x = o} = o \quad . \quad \text{Then} \quad C = \frac{ka^2}{\pi} \tag{32}$$

Substituting for C in Equation 31 gives

$$\frac{ka^2}{\pi} \, (1 + p^2)^{1/2} \quad = \frac{x^2}{2} + \frac{ka^2}{\pi} \tag{33}$$

Squaring the above equation on both sides and solving for p^2, there results

$$p^2 = \frac{\pi^2}{4k^2a^4} \, (x^2 + 2ka^2)^2 - 1 \tag{34}$$

Simplifying and solving for p gives

$$p = - \, (\frac{x^2}{\pi^2} + 2)^{1/2} \, \frac{x}{\pi} \tag{35}$$

** It is of interest to note that the solution y = constant no longer satisfies this equation.

Where the negative sign is chosen outside the radical to conform with the chosen coordinate system and

$$\frac{2ka^2}{\pi} = A_1^2$$

remembering that $p = dy/dx$, Equation 35 becomes

$$y = -\int \left(\frac{x^2}{A_1^2} + 2\right)^{1/2} \frac{x}{A_1} \, dx \qquad (36)$$

Upon integrating

$$y = \frac{-A_1}{3} \left(\frac{x^2}{A_1^2} + 2\right)^{3/2} + c_1 \qquad (37)$$

15. Consider, now, the boundary condition

$$y)_{x \to 0}^{0} = 0 \quad \text{where } \lambda \text{ is now considered } \Theta.$$

$$= \frac{-A_1}{3} \cdot 8 + c_1 \qquad (38)$$

or

$$c_1 = 0 + \frac{A_1}{3} \cdot 8 \qquad (39)$$

Substituting C_1 from Equation 41 into Equation 39 leads to the desired surface (Equation 40), which will produce a constant density of fragments, subjected to the assumptions made in paragraph 12.

$$y = \frac{-A_1}{3} \left(\left(\frac{x^2}{A_1^2} + 2\right)^{3/2} - 2^{3/2}\right) \quad \text{(See Fig 3)} \qquad (40)$$

where

$$A_1 = a\left[\frac{2k}{\pi}\right]^{1/2}$$

and

$$\eta = a \frac{\left(\frac{x^2}{A_1^2} + 2\right)^{1/2}}{A_1} \frac{x}{A_1} \qquad (41)$$

16. If for practical construction reasons (for example to allow space for a fuze well), a diameter (d) at the apex of the surface must be inoperative, it would be desirable to use the boundary $p = 0$ when $x = D/2$ when solving for the constant C_1 in order to cut down on the dead space on the target surface.

Using this boundary condition in conjunction with Equation 31, there results

$$C = \frac{A_1^2}{2} - \frac{D^2}{8} \tag{42}$$

Consequently $\quad \frac{A_1^2}{2}(1 + p^2)^{1/2} = \frac{x^2}{2} + \frac{A_1^2}{2} - \frac{D^2}{8}$ $\tag{43}$

and $\quad p^2 = (\frac{x^2}{A^2} + 1 - \frac{D^2}{4A^2})^2 - 1$

or $\quad y = \int_0^x \left[(x^2 + 1 - \frac{D^2}{4A^2})^2 - 1 \right]^{1/2} dx \tag{44}$

Equation 44 is the solution in integral form for this particular case. Given constants **D** and A, the desired surface is easily obtained by numerical integration.

Fragment Distribution for a Spherical Missile

17. The equation for a circle in polar coordinates is

$$\begin{aligned} y &= \cos \theta \\ x &= \sin \theta \end{aligned} \tag{45}$$

where r is a constant. When revolved about a diameter a spherical surface is generated.

It follows that $\quad y^1 = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\sin \theta}{\cos \theta} = \tan \theta \tag{46}$

and $\quad y^{11} = \dfrac{\frac{dy^1}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{-\sec^2\theta}{r \cos \theta} = \dfrac{-\sec^3\theta}{r} \tag{47}$

Substituting the values for x, y^1, and y^{11} in Equation 26, there results

$$K = \frac{M r^2}{a^2} \cos^3 \theta \tag{48}$$

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Letting $\dfrac{\mathcal{M} r^2}{a^2} = \lambda$

then
$$K = \lambda \cos^3 \theta \qquad\qquad (49)$$

and expressed as a dimensionless ratio

$$\frac{K}{\lambda} = \cos^3 \theta \quad \text{(See Fig 4)} \qquad\qquad (50)$$

General Solution for Axis Inclined to Target Surface

18. The solution obtained in paragraphs 8 to 11 is restricted in that the axis of the missile must bo perpendicular to the target surface. Obviously, this is not always the case, and it becomes of interest to find out how the fragment distribution varies when the missile axis is inclined to the target surface.

Fig. 2

12
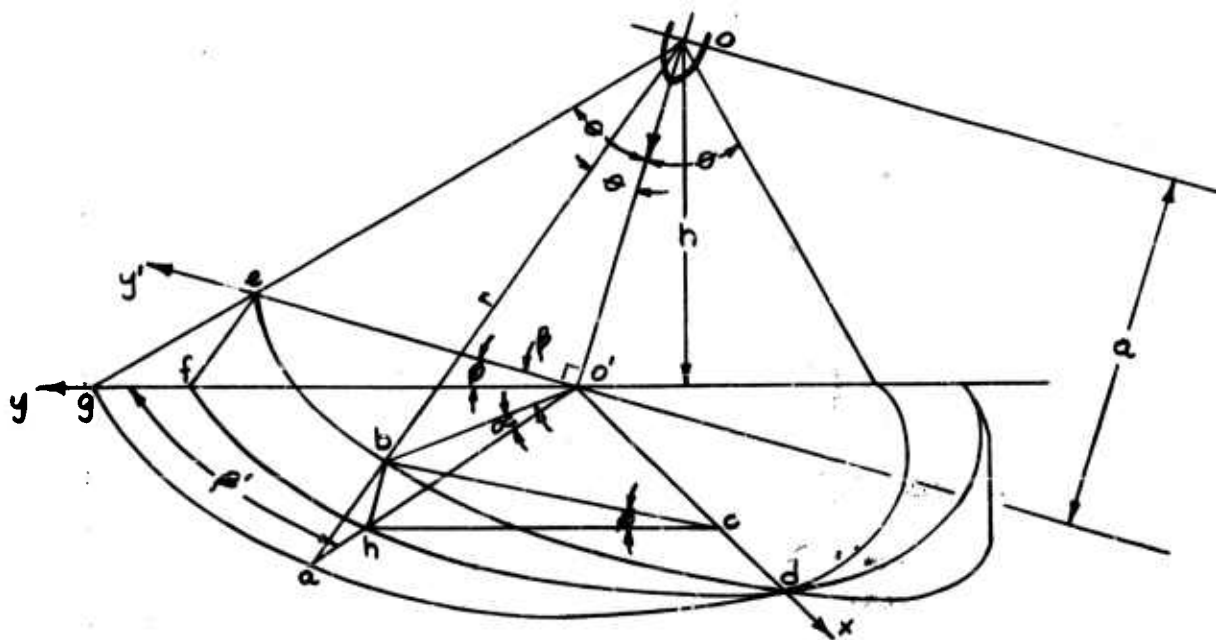
19. Consider Figure 2. The plane (eo'd) is perpendicular to line (oo'). It is desired to find the fragment density distribution on plane (go'd) which is inclined to line (oo'). The density K on plane (eo'd) at any point b is given by Equation 18, or by Equation 26 if the simplified solution is justified. The fragment density normal to R at point a is therefore

$$K_{n_a} = \frac{K}{\cos \theta} \frac{(ob)^2}{(oa)^2} \tag{51}$$

However, (ob) can be expressed as

$$(ob) = \frac{a}{\cos \theta} \tag{52}$$

and (oa) as

$$(oa) = \frac{a \sin \left(\frac{\pi}{2} + \alpha\right)}{\sin \left(\frac{\pi}{2} - \theta - \alpha\right)} = \frac{a \cos \alpha}{\cos (\theta + \alpha)} \tag{53}$$

K_{n_a} can therefore be expressed as

$$K_{n_a} = \frac{K \cos^2 (\theta + \alpha)}{\cos^3 \theta \cos^2 \alpha} \tag{54}$$

and the fragment density on plane (go'd) at point a as

$$K_a = \frac{K \cos^3 (\theta + \alpha)}{\cos^3 \theta \cos^2 \alpha} \tag{55}$$

It is more convenient to express the angle α as a function of ϕ, the angle the missile axis (OO') makes with the normal to the target surface (go'd), and β the angle between planes (ogo') and (oao'). This can be done as follows

$$\sin \phi = \frac{bh}{bc} \tag{56}$$

But

$$bc = a \tan \theta \cos \beta \tag{57}$$

and

$$bh = a \tan \theta \sin \alpha \tag{58}$$

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Consequently $\quad\quad\quad \sin\alpha = \sin\phi\cos\beta$ $\quad\quad\quad$ (59)

20. In order to locate the position of our new fragment distribution (o'a) in terms of (o'b) must also be determined. From Figure 2, it is observed that:

$$\frac{(o'a)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{(o'b)}{\sin\left(\frac{\pi}{2} - \theta - \alpha\right)} \quad\quad (60)$$

or $\quad\quad\quad \dfrac{(o'a)}{\cos\theta} = \dfrac{(o'b)}{\cos(\theta - \alpha)}$ $\quad\quad\quad$ (61)

The (o'b) is η of the general solution in paragraphs 8 to 11, so by letting (o'a) $= \eta'$

there results $\quad\quad\quad \eta' = \dfrac{\eta\cos\theta}{\cos(\theta - \alpha)}$ $\quad\quad\quad$ (62)

The fragment density and location can be determined on the inclined target plane by using Equations 55, 59, and 62, as long as the distribution is known on the plane normal to the missile axis.

21. Since the striking velocity of a fragment is a function of the length of path traveled, the length of the new fragment path is also of interest. It can be expressed as

$$(oa) = \frac{a\cos\alpha}{\cos(\theta - \alpha)} = \frac{h\cos\alpha}{\cos\phi\cos(\theta - \alpha)} \quad (63)$$

The length of the fragment path divided by the length of the path to the normal target plane is therefore

$$\frac{(oa)}{(ob)} = \frac{\cos\theta\cos\alpha}{\cos(\theta - \alpha)} \quad\quad (64)$$

22. Equations 55, 59, and 62 must be used to determine the effect of inclining the axis of the shape derived in paragraph 17. These equations are:

$$K_a = \frac{K\cos^3(\theta - \alpha)}{\cos^3\theta\cos^2\alpha}$$

$$\sin\alpha = \sin\phi\cos\beta \quad\quad\quad (65)$$

$$\eta' = \frac{\eta\cos\theta}{\cos(\theta - \alpha)}$$

where for this particular problem K is a constant.

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23. For a given angle ϕ, $K_{a/K}$ and η'/η could be plotted as functions of θ for various angles β. If desired, a set of tables or graphs could be drawn up in this manner. It would then be a rather simple process to compute the lethal area as a function of the angle of inclination.

Inclosures:
    1 - 2 Figures 3 and 4
    5 - 7 Appendices A, B, and C
           (Appendix B contains
             Tables 1 and 2)

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$$y/A_1 = 3 \left\{ \left[\frac{x^2}{A_1^2} + 2 \right]^{\frac{2}{3}} - 2^{\frac{2}{3}} \right\}$$

$$A_1 = a \left[\frac{2K}{N} \right]^{\frac{1}{2}}$$
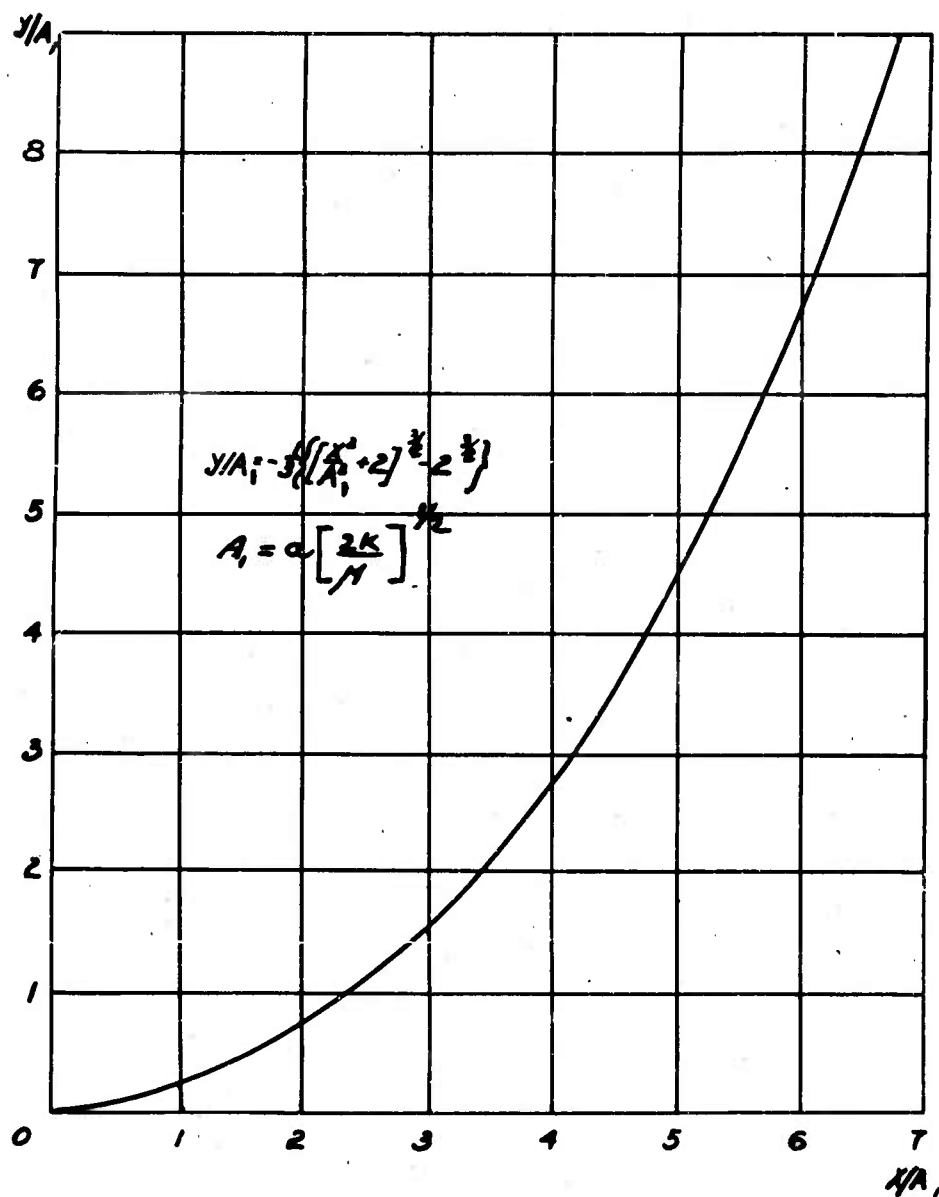
SURFACE CONTOUR OF THE MISSILE WHOSE FRAGMENTS
STRIKE THE GROUND WITH CONSTANT DENSITY
(Kent-Hitchcock Contour)

FIG. 3   CONFIDENTIAL

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K/λ

θ	$\cos\theta$	$\frac{K}{\lambda} = \cos^3\theta$
0	1.00000	1.00000
10	.98481	.95512
20	.93969	.82977
30	.86603	.64953
40	.76604	.44953
50	.64279	.26559
60	.50000	.12500
70	.34202	.04001
80	.17365	.00524
90	0	0

$$K = \frac{Nr^2}{a^2 \sec^3\theta} = \lambda \cos^3\theta$$

N = frag/unit area on missile surface
r = radius of missile
a = height of burst
K = frag/unit area on target surface

$\theta \longrightarrow$
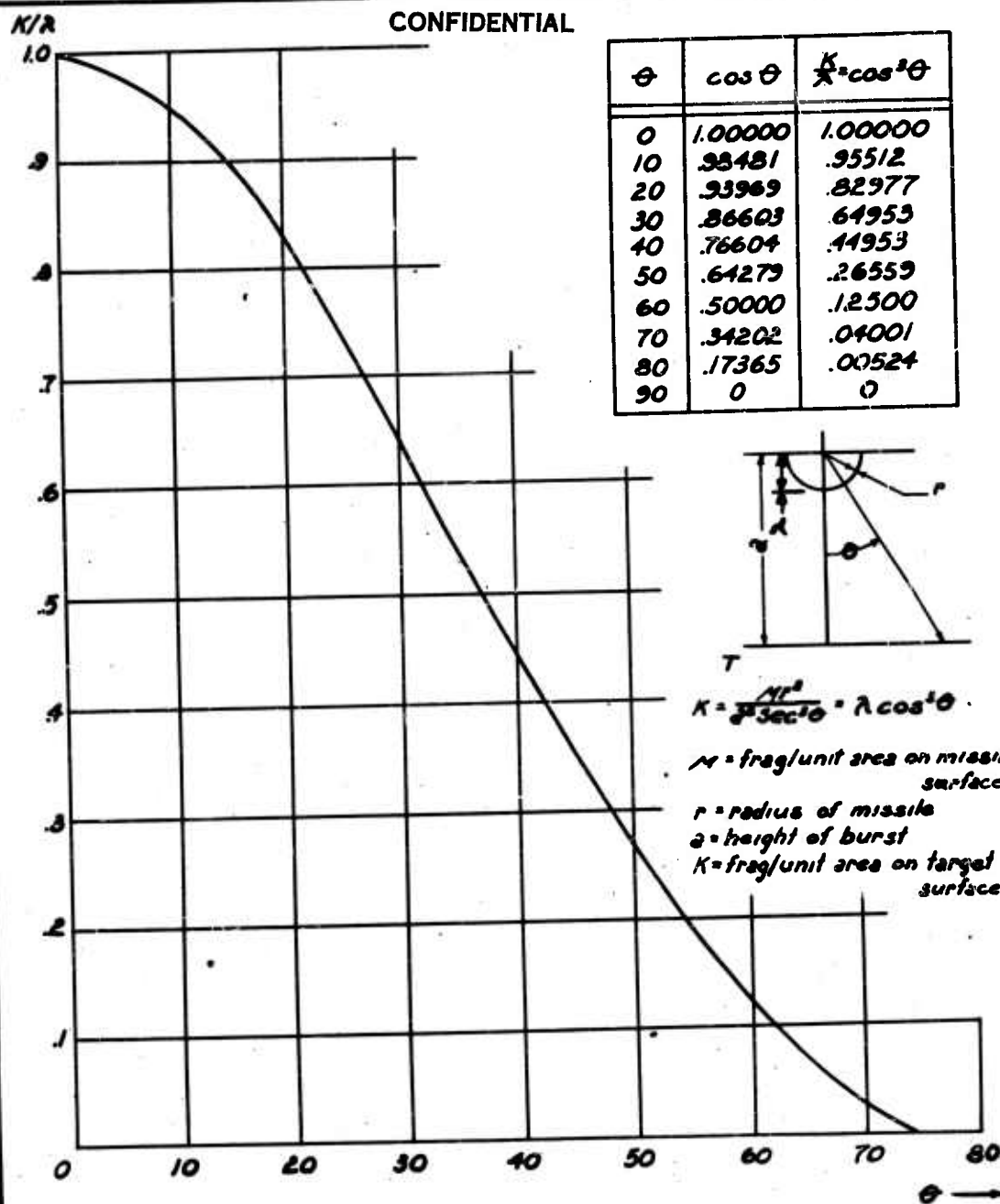
FRAGMENT DISTRIBUTION OF A SPHERICAL SHELL UPON A PLANE TARGET SURFACE.

FIG. 4   CONFIDENTIAL

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APPENDIX A

SURFACE THAT PRODUCES OPTIMUM LETHAL AREA UNDER CERTAIN ASSUMPTIONS
(This problem was solved by Mr. H. K. Weiss*)

The assumptions stated in the "Introduction" are assumed to hold, and in addition the axis of the missile is considered perpendicular to the target surface. The expected number of hits on a target can be expressed as:

$$E_h = \frac{M_\Theta}{r^2} A_\Theta \tag{1a}$$

where M_Θ = fragments per unit area per unit distance from the burst point, at angle Θ

r = distance the fragments travel to the target

A_Θ = exposed area of the target

The expected number of disabling hits is

$$E_k = E_h P_{hk} \tag{2a}$$

where P_{hk} is the conditional probability that a hit will disable the target. This is usually a function of the mass, velocity, and the presented area of a fragment. For mathematical simplicity, P_{hk} will be treated as a constant. This means that all the fragments are considered to have the same disabling potential.

By using the Poisson approximation, the probability of disabling the target is

$$P_k = 1 - e^{-E_k} \tag{3a}$$

The number of disabled targets is $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\sigma P_k \ dxdy \right]$ where σ is the number of targets per unit area. If σ is considered constant, it can be brought outside the integral. The integral term is then defined as the lethal area (A_L).

$$A_L = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_k \ dxdy \tag{4a}$$

* Weiss, H. K., op. cit.

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If the lethal area is multiplied by the density of the targets, therefore, the number of targets disabled will be obtained. Equation 4a can be expressed in polar coordinates as

$$A_L = 2\pi a^2 \int (1-e^{-E_k}) \sin\theta \sec^3\theta \, d\theta \qquad (5a)$$

where a = height of burst
           θ = angle between missile axis and fragment path

It is convenient at this point to make the following change of variables

$$W = \cos\theta$$
and $$dW = -\sin\theta \, d\theta$$

Equation 5a becomes

$$A_L = -2\pi a^2 \int (1-e^{-E_k}) \, W^{-3} \, dw \qquad (6a)$$

Furthermore $$E_k = E_h P_{hk} = \frac{M(\theta) \, A \, (\theta)}{r^2} P_{hk} \qquad (7a)$$

However $$r = a/\cos\theta = aw^{-1}$$

Therefore $$E_k = \frac{M(\theta) \, A \, (\theta) \, W^2}{a^2} P_{hk} \qquad (8a)$$

If it is also considered that the exposed area varies as the $\cos\theta$, as would a flat target, A_θ can be expressed as

$$A_\theta = A \cos\theta = Aw \qquad (9a)$$

Hence $$E_k = \frac{M_w \, AW^3}{a^2} P_{hk} \qquad (10a)$$

and $$A_L = -2\pi a^2 \int (1-e^{\frac{-M_w AW^3}{a^2} P_{hk}}) \, w^{-3} \, dw \qquad (11a)$$

A small increment of area at some position (W), over a band width (δW) is

$$(\delta A_L)_1 = -2\pi a^2 (1-e^{\frac{-M_w AW^3}{a^2} P_{hk}}) W^{-3} \, \delta W \qquad (12a)$$

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If M_W is now increased at the point (W), over the band width (δW) the increase in (δA_L), is

$$\delta(\delta A_L)_1 = \left[2\pi A\ P_{hk}\ e\ \frac{-M_W AW^3}{a^2}\ P_{hk} \right]_1 \delta W\ \delta M \qquad (13a)$$

but since the total number of fragments must not increase, δ M must be subtracted at some other point (W_2). At that point

$$\delta(\delta A_L)_2 = \left[2\pi A\ P_{hk}\ e\ \frac{-M_W AW^3}{a^2}\ P_{hk} \right]_2 \delta W\ \delta M \qquad (14a)$$

The necessary condition for M_W to be the surface that produces maximum lethal area is

$$\delta(\delta A_L)_1 - \delta(\delta A_L)_2 = 0 \qquad (15a)$$

Therefore

$$2\pi A\ P_{hk}\ e\ \frac{-M_W AW^3}{a^2}\ P_{hk} = C = \text{constant} \qquad (16a)$$

or

$$\frac{M_W AW^3 P_{hk}}{a^2} = Ln\left[\frac{2\pi A\ P_{hk}}{C} \right] \qquad (17a)$$

or

$$M_W = \left[\frac{a^2}{AP_{hk}}\ Ln\ \frac{(2\pi A\ P_{hk})}{C} \right] W^{-3} \qquad (18a)$$

Let

$$\frac{a^2}{A\ P_{hk}}\ Ln\ \frac{(2\pi A\ P_{hk})}{C} = A'$$

Then

$$M_W = A' W^{-3} \qquad (19a)$$

or

$$M_\Theta = A'\ \sec^3 \Theta \qquad (20a)$$

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Consider now the Kent-Hitchcock contour. It fulfilled the requirements that $K =$ constant where

$$K = \frac{Mxds}{\eta d\eta} \tag{21a}$$

or

$$K = \frac{M_\Theta \sin \Theta \, d\Theta}{\eta d\eta} \tag{22a}$$

But

$$\eta = a \tan \Theta$$

$$d\eta = a \sec^2 \Theta \, d\Theta$$

Therefore

$$K = \frac{M_\Theta \sin \Theta \, d\Theta}{a^2 \tan \Theta \sec^2 \Theta \, d\Theta} \tag{23a}$$

or M_Θ can be expressed as

$$M_\Theta = Ka^2 \sec^3 \Theta \tag{24a}$$

Equation 24a has the form of Equation 20a. Therefore, the Kent-Hitchcock contour is the contour that produces optimum lethal area under the restrictions of this particular problem.

APPENDIX B

APPLICATION OF ANALYSIS TO AN AERIAL EXPLODING BODY

Problem

To find the proper surface contour of an aerial exploding body that will produce a fragment distribution of constant density over the target surface T under the following restrictions:

1. Height (h) of the exploding body above target $=$ 20 ft

2. Desired density of the fragment distribution (k) over the target T $=$ 1.67 fragments/ft^2

3. Desired diameter (D) of target $=$ 40 ft

4. Fragment weight $=$ 2.5 grains

5. Surface area of fragment A_f $=$ .01175 in^2

6. Fragment distribution (μ) on body surface $=$

$$\frac{1}{.01175} = 85.106 \text{ fragments/in}^2$$

The surface which satisfies this problem is that of Equation 40. For simplicity, however, let $a = h$ and therefore $\lambda = 0$. Equation 40 would then reduce to

$$y = \frac{-A}{3} \left[(x^2 + 2)^{3/2} - (2)^{3/2} \right] \tag{1b}$$

In order to establish the boundary of the body in the x direction, an additional equation concerning the surface area of the body is needed. This is expressed mathematically as

$$A_s = 2 \int_0^x \pi (1 + (y^1)^2)^{1/2} \, x dx \tag{2b}$$

where from Equation 33 it can be shown that

$$(1 + (y^1)^2)^{1/2} = (\frac{x^2}{A^1} + 1) \tag{3b}$$

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Equation 2b can therefore be expressed as

$$A_s = 2 \int_0^{x_0} \left(\frac{x^3}{A^2} \neq x\right) \, dx \qquad (4b)$$

Upon integrating

$$A_s = \frac{(x_0^4 \neq x_0^2)}{2A^2} \qquad (5b)$$

Equation 5b can be expressed as

$$x_0^4 \neq 2A^2 \, x_0^2 - \frac{2A^2 A_s}{\pi} = 0 \qquad (6b)$$

where

$$A^2 = \frac{2ka^2}{\pi} = \frac{2(1.67)(20)^2}{85.106} = 15.698 \text{ in}^2$$

and in this particular problem

$$A_s = \frac{\pi D^2}{4} \cdot kA_f = \frac{(40)^2}{4} \cdot (1.67)(.01175) = 24.658 \text{ in}^2$$

Substituting these values in Equation 6b and solving for x_0 by using the quadratic formula, it is found that

$$x_0 = 2.550 \text{ in.}$$

By substitution of the value of A in Equation 40, the desired surface becomes

$$y = -1.3206 \left[(.06370x^2 \neq 2)^{3/2} - 2.8284 \right] \qquad (7b)$$

from which

$$y^1 = -.25237 \, x \, (.06370x^2 \neq 2)^{1/2} \qquad (8b)$$

and $y^{11} = -.01608 \, x^2 \, (.06370x^2 \neq 2)^{-1/2} - .25237 \, (.06370x^2$

$$\neq 2)^{1/2} \qquad (9b)$$

In order to find the exact density of fragment distribution on the target surface produced by Equation 7b, it is only necessary to substitute the above expressions in Equation 18. The results of

this substitution are presented in Table 1. It can be seen that the density of the fragment distribution is approximately 2% to 4% below the desired density of the fragment distribution. It is also seen that the target area is 3% larger than specified. This result suggests that the derived surface equation (1b) can be corrected to give improved results. It is proposed that the density of the fragment distribution be increased 3% by specifying that $K^1 = 1.03K$

Then $A^2 = \dfrac{2 (1.03\ ka^2}{} = (1.03)(15.698) = 16.169$

$A = 4.021$

Hence, the corrected equation for y becomes

$$y = -1.34030\ (.06185x^2 + 2)^{3/2} - 2.8284 \qquad (10b)$$

and $y^1 = -.24869x\ (.06185x^2 + 2)^{1/2} \qquad (11b)$

$y^{11} = -.01538x^2\ (.06185x^2 + 2)^{-1/2} -.24869$

$$(.06185x^2 + 2)^{1/2} \qquad (12b)$$

Table 2 gives the improved results. The density of the fragment distribution now varies from + 1/2% to - 1%, while the target area is in error by less than 1/2%

APPENDIX C

COMPARISON OF A SPHERE AND THE KENT-HITCHCOCK CONTOUR FOR LETHAL AREA

From Appendix A it was shown that the Kent-Hitchcock contour produced the maximum lethal area for the type of problem considered. It remains of interest however to find how much greater this lethal area is than a conventional sphere. Consider the problem where the target radius (L) equals the burst height (a).

The distribution of fragments from a sphere is

$$K_s = K_o \cos^3 \theta \tag{1c}$$
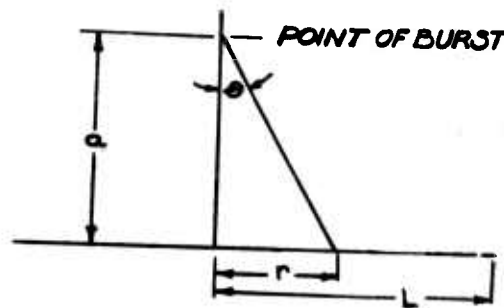
Fig 5

The number of fragments over the target surface of radius L is

$$N_s = 2\pi K_o \int_0^L \cos^3 \theta \, r dr = 2\pi K_o \int_0^L \frac{L^3 \, r dr}{(r^2 + L^2)^{3/2}} \tag{2c}$$

Where K_o is a constant that is a function of the burst height, radius of the sphere, and fragments per unit area on the sphere surface.

The number of fragments over the target surface of radius L from the Kent-Hitchcock contour is simply

$$N_{kh} = K_{kh} \pi L^2 \tag{3c}$$

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TABLE 1 – Fragment Distribution Produced by Equation 7b

	0	0.4	0.8	1.2	1.6	2.0	2.4	2.55
x	0	.4	.8	1.2	1.6	2.0	2.4	2.55
x^2	0	.15000	.64000	1.4400	2.5600	4.0000	5.7600	6.5030
$(.0637x^2/2)$	2.0000	2.0102	2.0405	2.0917	2.1631	2.2480	2.3669	2.4142
$(.0637x^2+1)^{1/2}$	1.4142	1.4180	1.4290	1.4463	1.4710	1.5010	1.5380	1.5540
$(\ldots)^{3/2}$	2.8284	2.8512	2.9181	3.0253	3.1830	3.3818	3.6380	3.7528
$(\ldots)^{-1/2}$	.70711	.70521	.69979	.69141	.67980	.66322	.65019	.64350
y	0	.0301	.1185	.2600	.4683	.7308	1.0692	1.220 #
y'	0	.1431	.2885	.4380	.5940	.7576	.9315	1.0000
$x^2(\ldots)^{-1/2}$	0	.1128	.4479	.9956	1.7403	2.6649	3.7451	4.1847
y''	.3569	.3597	.3678	.3810	.3992	.4217	.4484	.4595
y'	0	34.344	69.240	105.12	142.56	181.82	223.56	240.00
y''	0	.004	.034	.114	.278	.554	.996	1.221
η	0	34.748	70.074	106.43	144.44	184.37	226.95	243.77
$(y')^2$	0	.0205	.0832	.1918	.3528	.5740	.8677	1.0000
$(y')^2$	85.656	86.328	88.272	91.440	95.908	101.208	107.616	110.280
y'	0	.011	.044	.099	.187	.308	.479	.561
ρ	86.656	87.359	89.399	92.731	97.348	103.090	109.963	112.841
$(1+(y')^2)^{1/2}$	1.000	1.010	1.041	1.092	1.163	1.255	1.367	1.4142
$\int x(1+(y')^2)^{1/2}$	0	34.383	70.876	111.52	158.37	213.62	279.22	306.91
η^2		3035.5	6264.6	9849.4	14061	19007	24956	27507
K fr/in^2		.01133	.01131	.01130	.01126	.01124	.01119	.01116
K fr/ft^2		1.632	1.629	1.627	1.621	1.619	1.611	1.607

TABLE 2 - Fragment Distribution Produced by Equation 7B (Corrected)

	0	0.4	0.8	1.2	1.6	2.0	2.4	2.55
x	0	0.4	0.8	1.2	1.6	2.0	2.4	2.55
x^2	0	0.1600	0.6400	1.440	2.560	4.000	5.760	6.503
$(.06185x^2 f/2)$	2.0000	2.0099	2.0396	2.0891	2.1583	2.2474	2.3563	2.4022
$(.06185x^2 f)^{1/2}$	1.4142	1.4176	1.4281	1.4454	1.4692	1.4992	1.5350	1.5500
$(.06185x^2 f)^{3/2}$	2.8284	2.8488	2.9127	3.01970	3.1713	3.3696	3.6168	3.7239
$(.06185x^2 f)^{-1/2}$	.70711	.70541	.70023	.69185	.68064	.66702	.65146	.64516
y	0	.0273	.1130	.2564	.4596	.7254	1.0567	1.200
y^1	0	.1410	.2841	.4313	.5846	.7457	.9162	.9829
y^{11}	−0	.1129	.4481	.9963	1.7424	2.6681	3.7524	4.1955
$x^2(.06185x^2 f)^{-1/2}$	.3517	.3543	.3620	.3748	.3922	.4139	.4395	.4500
xy^1	0	33.840	68.184	103.51	140.30	178.97	219.89	235.90
xy^{11}	0	.004	.032	.11	.27	.54	.97	1.18
η	0	34.244	69.016	104.82	142.17	181.51	223.26	239.63
$(y^1)^2$	−0	.0198	.0807	.1860	.3418	.5561	.8394	.9661
xy^{11}	84.408	85.032	86.880	89.952	94.128	99.336	105.48	108.00
xy^{11}	0	.009	.041	.096	.180	.300	.46	.54
β	85.408	86.060	88.002	91.234	95.650	101.19	107.78	110.51
$(1+(y^1)^2)^{1/2}$	1.000	1.009	1.039	1.089	1.158	1.247	1.355	1.402
$x(1+(y^1)^2)^{-1/2}$	0	34.349	70.740	111.22	157.68	212.25	276.76	304.26
η/β	0	2947.0	6073.5	9563.1	13599	18367	24063	26482
K fr/in²		.01166	.01165	.01163	.01159	.01156	.01150	.01149
K fr/ft²		1.679	1.678	1.675	1.669	1.665	1.656	1.655

To make a comparison of lethal area, N_{kh} is assumed to be equal to N_s. By equating these two expressions, it is found that

$$K_0 = \frac{K_{kh}}{(2-\sqrt{2})} = 1.706 K_{kh} \tag{4c}$$

Lethal area can be expressed as (Equation 6a)

$$A_L = -2\pi a^2 \int (1-e^{-Ek}) W^{-3} dw \tag{5c}$$

where

$$Ek = K A_\theta P_{hk} \tag{6c}$$

Consider the $A_\theta = A \cos \theta = AW$. The lethal area for a sphere is

$$A_{L/s} = 2\pi A^2 \int (1-e^{-1.706 AK_{kh}P_{hk}W^4}) W^{-3} dW \tag{7c}$$

since

$$K = K_0 W^3 = 1.706 K_{kh} W^3$$

where

$$AK_{kh} P_{hk} = \text{constant}$$

The lethal area for the Kent-Hitchcock contour can be expressed as

$$A_L \Big|_{KH} = 2\pi a^2 \int (1-e^{-AK_{kh}P_{hk}W}) W^{-3} dw \tag{8c}$$

Assume $AK_{kh} P_{hk} = 2.5$ (This is a fair value for a sample problem)

Then

$$\frac{A_L}{2\pi a^2} \Big|_s = \int (1-e^{-4.26W^4}) W^{-3} dw \tag{9c}$$

$$\frac{A_L}{2\pi a^2} \Big|_{kh} = \int (1-e^{-2.5W}) W^{-3} dw \tag{10c}$$

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Further assuming that the limit of the target area corresponds to $\Theta = 45°$, then the limits on the above integration are 1 to .707 since $W = \cos \Theta$. Carrying out the above integration numerically, there results

$$\frac{A_{L/kh}}{A_{L/s}} = 1.166 \tag{11c}$$

That is, the Kent-Hitchcock contour gives a lethal area that is 16.6% greater than a sphere, assuming that both surfaces contain the same number of fragments.

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